N-dimensional Abelian sandpile model with nearest-neighbor toppling

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(Received 1 July 1992)

We have derived an exact expression for the determinant of the toppling matrix of the Abelian sandpile model in any finite dimension with open boundary conditions. The result can also apply to the semiopen boundary cases in certain limiting cases. An analytic result can also be obtained in the thermodynamic limit where the grid size tends to infinite: namely, the total number of system configurations in the self-organized critical state follows a power law as the grid size increases. Therefore, under a uniform but random particle addition, $1/f^2$ instead of $1/f$ scaling is observed. The relation with the site percolation problem is also discussed.

PACS number(s): 05.40.+j, 64.60.Fr

I. INTRODUCTION

The concept of self-organized criticality, which was recently introduced by Bak, Tang, and Wiesenfeld [1], has been used to study quite a number of physical phenomena, including the sandpile [1,2] and earthquakes [3]. In particular, it was claimed first by Bak *et al.* that under a random and uniform particle addition to the pile (with dimension larger than 1) a $1/f$ power spectrum in avalanche size would result. However, it was later argued by various authors, using both statistical and numerical methods, that in the thermodynamic limit, the pile should result in a $1/f^2$ power spectrum instead [4], whenever the dimension of the pile was less than 6. In this paper, we are going to first show that the total number of self-organized critical states (which is equal to $\det\Delta$ [5]) scales as a power law, as the total number of grids in the system increases for any finite number of dimensions. Thus, under a uniform and random particle addition to the system, a $1/f^2$ power spectrum is a natural consequence. In Sec. II, we shall first review some facts about symmetric and Toeplitz matrices. Then we shall apply the result to calculate the total number of self-organized critical states in various system configurations with different boundary conditions and different dimensions. We shall also argue the form of the power spectrum of avalanche size. Relation to percolation is also discussed. Finally, a brief summary is presented in Sec. III.

II. CALCULATION OF DET Δ

The N-dimensional Abelian sandpile model (ASM) with nearest-neighbor toppling is defined as follows: construct an N-dimensional simple-cubic lattice and use all its lattice points as our grid. Assign an integer called the local height to each of the grid points. Whenever a local height is greater than a certain prescribed value, the corresponding grid point is said to be unstable. It will then rearrange itself by distributing $2N$ of its local height equally to its nearest neighbor, which is termed toppling. This process is repeated until all the grid points of the system become stable again and the entire process is called an avalanche. So we may think of the local height as the amount of particles a grid point holds. If we label the grid points by i, then a (toppling) matrix Δ can be formed in the following way: Δ_{ii} is the number of particles we have to remove from i and possibly have to transport to other grid points whenever i becomes unstable, while Δ_{ij} is minus the amount of particles received by j whenever i topples for $j\neq i$. Thus the toppling matrix is well defined once the boundary conditions of the system are specified. In particular, we shall consider the following two important types of boundary conditions: a boundary hypersurface of the system is said to be open, provided that every particle that is moved out of the system from this hypersurface during an avalanche cannot enter the system once again; on the other hand, it is said to be closed if no particle can move out of this boundary in any avalanche. Thus a system is said to be having an open boundary, provided that all its $2N$ hypersurfaces are open, and it is said to have a semiopen boundary as long as N of them (which intersect at a point) are open and N of them are closed [6]. As pointed out by Dhar, if we repreatedly add a unit amount of particles randomly and uniformly over the grid points at a slow enough rate, the system will eventually evolve to a steady state called the self-organized critical state, whose total number of configurations in these kinds of states is given by the determinant of its toppling matrix Δ [5], and now we are going to discuss some facts on symmetric and Toeplitz matrices first. After that we are going to apply the result to calculate the determinant and the inverse of the toppling matrices for various Abelian sandpile models with nearest-neighbor toppling.

A. Facts on symmetric and Toeplitz matrices

Fact 1. Every symmetric matrix is diagonalizable [7]. In particular, if we consider the $n\times n$ tridiagonal matrix $\mathbf{T}_1^n$ with $\mathbf{T}^n_{1_{ii}}=a$ (≥ 2) for all i, $\mathbf{T}^n_{1_{ij}}=-1$ whenever

$|i-j|=1$; that is,

$$\mathbf{T}_1^n=\begin{bmatrix} a & -1 & & & \\ -1 & a & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & a & -1 \\ & & & -1 & a \end{bmatrix} . \tag{1}$$

Fact 2. The eigenvalues of the $n\times n$ matrix $\mathbf{T}_1^n$ are $a-2\cos(k\pi/n+1)$ with $k=1,2,\ldots,n$ [8]. Moreover, the corresponding normalized eigenvectors y_i are $\sqrt{2/(n+1)}\sin[ik\pi/(n+1)]$.

Proof. Suppose $f_n(\lambda)$ is the characteristic polynomial of $\mathbf{T}_1^n$, then it satisfies the recurrence relation $f_n=(a-\lambda)f_{n-1}-f_{n-2}$. From the initial conditions, it is clear that $f_n(\lambda)=\sin(n+1)x/\sin x$ where $\lambda=a-2\cos x$. Therefore the eigenvalues are obvious. The corresponding eigenvectors can be verified directly.

Actually, if we define D_n as the determinant of $\mathbf{T}_1^n$, then D_n forms a Sturm sequence which can be used to estimate the distribution of eigenvalues of $\mathbf{T}_1^n$ [7,9]. Moreover, the location of the eigenvalues of a matrix is also bounded by a finite union of discs in C, the set of all complex numbers [7,9,10]. Furthermore, the determinant of $\mathbf{T}_1^n$ given by

$$\det\mathbf{T}_1^n=\begin{cases} n+1 & \text{if } a=2 \\ \dfrac{\mu_2^{n+1}-\mu_1^{n+1}}{\mu_2-\mu_1} & \text{otherwise,} \end{cases} \tag{2}$$

where μ_i are the characteristic roots of $\mu^2-a\mu-1=0$. So by combining Fact 2 and Eq. (2), we have

$$\prod_{k=1}^{n-1}\left[a-2\cos\frac{k\pi}{n}\right]=\frac{\mu_2^n-\mu_1^n}{\mu_2-\mu_1} . \tag{3}$$

In addition, if we consider another matrix $\mathbf{S}_1^n$, which equals $\mathbf{T}_1^n$ except that the first diagonal element $\mathbf{S}_{1_{11}}^n=b(<a)$ instead of a, that is,

$$\mathbf{S}_1^n=\begin{bmatrix} b & -1 & & & \\ -1 & a & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & a & -1 \\ & & & -1 & a \end{bmatrix} , \tag{4}$$

then we have

$$\det\mathbf{S}_1^n=\begin{cases} b & \text{if } a=2 \quad n=1 \\ n(b-1)+1 & \text{if } a=2, \quad n>1 \\ \dfrac{b(\mu_2^n-\mu_1^n)-(\mu_2^{n-1}-\mu_1^{n-1})}{\mu_2-\mu_1} & \text{otherwise .} \end{cases} \tag{5}$$

Besides, by means of the Sturm sequence, as just discussed before together with the Gerschgorin theorem [10], it is easy to see that the total number of eigenvalues of $\mathbf{S}_1^n$ in the open intervals $(a-2,a)$ and $(a,a+2)$ are the same. In fact, we can say more about the distribution of eigenvalues of $\mathbf{S}_1^n$, the proof of which can be found in Appendix A.

Theorem 1. As $n\rightarrow\infty$, the eigenvalues of $\mathbf{S}_1^n$ in the interval $(a-2,a+2)$ shall follow the distribution $D(x)=\pi\cos[\pi(x-a)/4]/8$.

Actually, the eigenvalues of $\mathbf{S}_1$ are also known in some special but important cases including the following.

Fact 3. The eigenvalues of $\mathbf{S}_1^n$ are $a-2\cos[(2k+1)\pi/(2n+1)]$ with $k=0,1,\ldots,n-1$ whenever $b=a-1$.

Proof. Similar to Fact 2 and Eq. (A1), $f_n(\lambda)=[\sin(n+1)x-\sin nx]/\sin x$ and so the eigenvalues can be computed easily.

Now we can define the first generation Toeplitz ($\mathbf{T}_1^n$) and tridiagonal symmetric matrices ($\mathbf{S}_1^n$) as the Toeplitz and tridiagonal symmetric matrices, as defined above respectively. Also the mth generation Toeplitz matrix $\mathbf{T}_m^n$ is made up of an $n\times n$ block matrix of the form

$$\mathbf{T}_m^n=\begin{bmatrix} \mathbf{A} & -\mathbf{I} & & & \\ -\mathbf{I} & \mathbf{A} & -\mathbf{I} & & \\ & \ddots & \ddots & \ddots & \\ & & -\mathbf{I} & \mathbf{A} & -\mathbf{I} \\ & & & -\mathbf{I} & \mathbf{A} \end{bmatrix} \tag{6}$$

where $\mathbf{A}$ is an $(m-1)$th generation Toeplitz matrix and $\mathbf{I}$ is the identity matrix. Similarly, the mth generation tridiagonal symmetric matrix $\mathbf{S}_m^n$ is made up of a block matrix that is similar to the one above except that the first element in the diagonal of the block matrix need not be the same as the rest of them. Then we have the following observation.

Theorem 2. The eigenvalues of $\mathbf{T}_m^n$ and $\mathbf{S}_m^n$ are real and may possibly be degenerate.

Proof. Let us consider only the mth generation Toeplitz matrix; otherwise the proof is similar. We prove this by induction on m. When $m=1$, the result is just given by Fact 2. Consider an mth generation Toeplitz matrix $\mathbf{T}_m^n$ that is generated by an $(m-1)$th generation matrix $\mathbf{A}$. By an induction assumption, $\mathbf{A}$ is diagonalizable, so that we can find a unitary matrix $\mathbf{U}$ such that $\mathbf{U}\mathbf{A}\mathbf{U}^{-1}$ is diagonalized. We can construct a block diagonal unitary matrix $\mathbf{V}$ with all its diagonal elements equal $\mathbf{U}$. Then $\mathbf{V}T_m^n\mathbf{V}^{-1}$ is block tridiagonalized and all its off-diagonal elements are zero, except at each of the two off-diagonal lines. By relabeling the columns and rows, the matrix can be converted into a block diagonal matrix where each of the blocks (possibly identical) is a tridiagonal matrix in the form of $\mathbf{T}_1^n$. Therefore, it is proved.

Note that the above proof also tells us how to find the eigenvalues of an mth generation Toeplitz matrix. In addition, it is clear that the eigenvalues for a higher generation matrix $\mathbf{T}_m^n$ or $\mathbf{S}_m^n$ are highly degenerate.

A similar idea has already been used recently by Markošová and Markoš in the calculation of attractor periods of deterministic sandpiles with open boundary conditions by means of tensors and field-theoretic methods and notations. The relation with the lattice Green's function is also mentioned [11].

B. One-dimensional models

With the aid of the above observations, we are now going to calculate the value of $\det\Delta$ for one-dimensional systems. From the discussion in Sec. I, the toppling matrix of the one-dimensional model with open boundary conditions (that is, where particles can freely go away from the system through the two ends at the time of toppling) is just a tridiagonal Toeplitz matrix with diagonal elements equal to 2 and off-diagonal elements equal to -1 [1,5]. So from Eq. (2), $\det\Delta$ is just $n+1$ where n is the number of grid points of the system. In other words, the total number of self-organized critical configurations of the system grow linearly with the number of grid points in the system. So under a uniform and random particle addition, the $1/f^2$ power spectrum in the avalanche size is observed [4,6].

In a similar way, for a one-dimensional semiopen boundary system, where particles can only go out of the system from one end at the time of toppling, the toppling matrix is in the form of $\mathbf{S}_1^n$ with $a=2$ and $b=1$. So from Eq. (5), there is one and only one self-organized critical configuration for the system which is independent of the number of grid points n. In other words, no matter where you drop the sand, it will simply slide through the system and dissipate in the open side of the system boundary. Thus the situation is exactly the same as the one-dimensional model studied by Bak, Tang, and Wiesenfeld [1]. So under a uniform and random particle addition, $1/f^2$ scaling in its power spectrum is observed [1,4,6].

Actually in both models the two-point correlation function $G_{ij}=\Delta_{ij}^{-1}$, which is the probability of toppling occuring at site j given that a particle is added at site i, will never die out as the distance between i and j in the system increases [5]. It is this kind of strong correlation in the self-organized critical states of the system that leads to the $1/f^2$ power law of avalanche size. However, if the system is allowed to remove more than two particles each time it topples (and hence the particle number is not conserved during an avalanche), from Eq. (2) or Eq. (5) the total number of self-organized critical states will grow exponentially with the number of grid points. This implies an exponential decay in the two-point correlation function, and hence a $1/(f+a)^2$ power spectrum is observed for some $a>0$ [12]. In other words, the scaling in the power spectrum is a direct consequence of the existence of a particle conservation law in the model. Further discussions on the conservation law can be found elsewhere [13].

C. Two-dimensional models

After looking at the trivial one-dimensional models, we quickly turn to their more interesting two-dimensional counterparts. From the above discussions, we know that all the diagonal elements of Δ for a two-dimensional model with open boundary conditions (where particles can go out in all the four boundary surfaces) are 4. Also, $\Delta_{ij}=-1$ if and only if i and j are neighboring grid points. Therefore Δ is a second-generation Toeplitz matrix. Thus if the grid size is $n\times m$, then from Theorem 2 and Fact 2 the eigenvalues of Δ are $4-2\cos[j\pi/(n+1)]-2\cos[k\pi/(m+1)]$ with $j=1,2,\ldots,n$ and $k=1,2,\ldots,m$. From Eq. (3), $\det\Delta$ is given by

$$\det\Delta=\prod_{j=1}^{n}\prod_{k=1}^{m}\left[4-2\cos\frac{j\pi}{n+1}-2\cos\frac{k\pi}{m+1}\right]$$

$$=4^{mn}\prod_{j=1}^{n}\prod_{k=1}^{m}\left[\sin^2\frac{j\pi}{2(n+1)}+\sin^2\frac{k\pi}{2(m+1)}\right]$$

$$=\prod_{j=1}^{n}\frac{\mu_{2j}^{m+1}-\mu_{1j}^{m+1}}{\mu_{2j}-\mu_{1j}}\ , \tag{7}$$

where μ_{1j} and μ_{2j} are roots of the characteristic equation $\lambda^2-\{4-2\cos[j\pi/(n+1)]\}\lambda+1=0$ for every j. A similar equation can also be obtained for nonconservative systems.

In the thermodynamic limit where $m,n\to\infty$, $\det\Delta\to\zeta^{mn}$, where ζ is given by

$$\ln\frac{\zeta}{4}=\frac{4}{\pi^2}\int_0^{\pi/2}\int_0^{\pi/2}\ln[\sin^2x+\sin^2y]dx\,dy\ , \tag{8}$$

which implies that $\zeta\approx3.212$. That is, an average of ζ distinct states can be found in each of the local grid points once the system has evolved to its self-organized critical configuration. Although the phase-space volume of the system in the self-organized critical state is rather large, it does not necessarily imply an exponential decay in the two-point correlation function G_{ij}. Because one of the eigenvalues of the system is $4-2\cos[\pi/(n+1)]-2\cos[\pi/(m+1)]$, it will tend to 0 as $m,n\to\infty$; thus some of the elements in $G(=\Delta^{-1})$ may be quite large. So we have to calculate G_{ij} explicitly, at least in some special cases. In fact, we shall prove in Appendix B that $G_{i,j;k,l}$, the probability of having an avalanche in grid (k,l) given a particle is added to the site (i,j), is given by

$$G_{i,j;k,l}=\left[\frac{2}{n+2}\right]^2\sum_{a,b=1}^{n}\frac{\sin\frac{ia\pi}{n+1}\sin\frac{ka\pi}{n+1}\sin\frac{jb\pi}{n+1}\sin\frac{lb\pi}{n+1}}{4-2\cos\frac{a\pi}{n+1}-2\cos\frac{b\pi}{n+1}} \tag{9}$$

whenever $m=n$. So in the thermodynamic limit, some of the $G_{i,j;k,l}$ may become rather large due to the fact that the denominator approaches zero. By means of product and sum formulas in trigonometry, it is not difficult to see that $G_{i,j;k,l}$ satisfies the following sum rule:

$$G_{i,j;k,l}=\sum_{p=1}^{k}\sum_{q=1}^{l}G_{1,1;i-k+2p-1,j-l+2q-1} \quad \text{for all } i,j,k,l\ , \qquad (10)$$

where we define $G_{1,1;k,l}=-G_{1,1;-k,l}$ whenever $k\leq 0$ and similarly for l. By means of the fact that $G_{i,j;k,l}=G_{n+1-i,n+1-j;n+1-k,n+1-l}$ and Eq. (10), we can calculate the entire two-point correlation function by knowing all the values of $G_{1,1;k,l}$. Equation (10) rules out the possibility that the two-point correlation function scales like a power law as the separation between two sites increases. The reason is that if $G_{i,j;k,l}$ scales like $1/s^{\alpha}$, where s is the distance between (i,j) and (k,l), then Eq. (10) implies that $G_{i,j;k,l}$ will also scale as $1/s^{\alpha-1}$ for sufficiently large s, which is impossible. Besides, we have

$$\sum_{k.l=1}^{n}G_{1,1;k,l}=\left[\frac{2}{n+1}\right]^2\sum_{a,b=0}^{\left[\frac{n-1}{2}\right]}\frac{\left[1+\cos\frac{(2a+1)\pi}{n+1}\right]\left[1+\cos\frac{(2b+1)\pi}{n+1}\right]}{4-2\cos\frac{(2a+1)\pi}{n+1}-2\cos\frac{(2b+1)\pi}{n+1}}\approx\frac{8}{\pi}\ln\left[\frac{n+1}{\pi}\right]\ , \qquad (11)$$

which diverges like ln n in the thermodynamic limit. We use $[x]$ to denote the integral part of x. Therefore, the two-point correlation function cannot die out exponentially. Also $G_{1,1;k.l}-G_{1,1;k+2,l}=G_{2,1;k+2,l}>0$, which implies that G is an overall straightly decreasing function of k and l. From Eq. (9), we know that as $m=n\to\infty$, G is still analytic. So we may expand $G_{1,1;k,l}$ in terms of a Laurent series with variables k and l. The above argument forces this Laurent series to be analytic. The divergence of Eq. (11) therefore implies that G is algebraic in the sense that $G_{i,j;k,l}$ can only decay algebraically as the separation between (i,j) and (k,l) increases. In other words, the power spectrum in avalanche size will eventually scale as $1/f^2$ instead of $1/f$, as that which was proposed previously [1,4,14]. Although the above discussion is based on the assumption that $m=n$, we expect the same result to also hold in the general case where m and $n\to\infty$ separately. The physical reason is simple: the behavior of this kind of system in the thermodynamic limit must be well defined and unique.

Physically, the appearance of a long-range correlation at the expense of an infinite number of self-organized critical states is not self-contradictory. When the system has evolved to one of its self-organized critical configurations, none of its local grid points can afford an addition of four particles at the same time without being unstable. Also those grid points located just adjacent to an avalanche cluster are used to absorb all the particles coming out from this avalanche cluster. For a sufficiently small avalanche, the average number of particles coming out of the avalanche cluster is not very large, and hence the probability of preventing it from further growth is quite high. But for very large avalanches, the total number of particles that these boundary sites have to receive can easily exceed four. Thus, the only possibility is that the avalanche will continue to grow until it reaches the system boundary and will dissipate there.

In the event that the system is dissipative, the diagonal element of Δ is greater than 4. Therefore as m and $n\to\infty$, none of the eigenvalues of Δ will tend to zero. We expect an exponential type of decrease in the correlation length. This point can be further strengthened by the fact that $\sum_{k,l=1}^{n}G_{1,1;k,l}<1/(x-4)$ is finite, where $x\,(>4)$ is the number of particles toppled each time. At the other extreme, if the system is creative, that is, new particles are introduced every time a site topples, then some of the eigenvalues of Δ become negative and even zero. So in the thermodynamic limit, the two-point correlation function becomes an ill-defined concept. In this respect, the existence of a particle conservation law is of great importance to the exhibition of self-organized criticality for this kind of sandpile model. Further discussions on the importance of conservation laws can be found elsewhere [13,14].

Now let us look into the problem from the viewpoint of (site) percolation [16]: suppose we have an ASM model on an $m\times n$ square lattice such that in the thermodynamic limit the total number of self-organized critical states scales as $1/p^{mn}$ with $p\leq 1$. Then for each single particle added to an arbitrary site, the average probability of that site becoming unstable and hence inducing possibly a series of topplings is p. So let us compare this with the problem of site percolation; namely, every grid point of the square lattice has precisely a probability p of being occupied. We can obtain, at least in principle, the distribution function for the percolating animals (that is, the distribution of the size of connected occupied sites) for this problem. We may generate such a distribution in the following way: randomly choose a site on the lattice and ask if that site is occupied. If not, then there is no percolating animal associated with that site. If the answer is positive, then we look around for its four neighboring sites and ask the same question again of them. By repeating this procedure, we know the size of the percolating animal in question. After sufficiently large statistics, we get the idea of how such a percolating size is distributed. Note that the probability p of a site being occupied is independent of all the other sites.

However, if we compare this with how we determine the avalanche size, the situation is different. The probability of a given site being stable or not depends on the total number of particles added or toppled to the site. So when we add a particle to a site, the probability of that site being unstable is p. But whenever there is a large avalanche, due to the two-dimensional nature of the system, which allows an interconnected flow of particles via toppling using different paths to the same point at the same time, the average number of particles received by some of the sites can be far greater than 1. So the probability of some of the sites (possibly near the point where we add a particle to trigger the avalanche) being toppled is far greater than p. So if we can increase the value of p from 0 up in both the percolating and self-organized critical systems at the same time, the distribution avalanche size for the self-organized critical system will go critical before that of the percolating animal for the corresponding percolation system. Therefore, if we use $p=1/\zeta$, which is found in the two-dimensional ASM model, as the probability of occupation of sites in the corresponding percolation problem, that system must be supercritical in the sense that the probability of obtaining a percolating animal with sufficiently large size is exponentially small. That is to say, the value of $p=1/\zeta$ sets a lower bound for the percolation threshold (p_c) in the corresponding site percolation problem as long as the sandpile model is critical or subcritical. It should be noted that we have a further degree of freedom here for finding the lower bound of the site percolation threshold: namely, the sandpile model itself. For different sandpile models with the same corresponding site percolation, the values of p may be different. Therefore, we can vary the rules of the sandpile model so as to locate a better lower bound for the p_c. Further discussions on percolation on different lattices can be found elsewhere [15]. In this two-dimensional case, we expect the $1/\zeta \approx 0.311$ is a lower bound for the two-dimensional site percolation problem, which is consistent with the simulation results [16].

Let us now go on to the problem of a two-dimensional ASM with semiopen boundary conditions on an $m \times n$ grid. (That is, particles can only go out from two adjacent of the four boundary surfaces.) Similar to the argument early in this subsection, the toppling matrix Δ for this model is a second-generation tridiagonal matrix. Although the exact eigenvalues, eigenvectors, and hence the matrix inverse of Δ is unknown, Theorem 1 ensures that with the exception of possibly $m+n-1$ eigenvalues, the distribution of the rest of them will asymptotically tend to the corresponding two-dimensional model with open boundary conditions. Moreover, it is straightforward to see that $\det\Delta$ will scale as $\zeta^{(m-1)(n-1)}\beta^{m+n-1}$ with $\beta<\zeta$. So with the same grid, the number of self-organized critical states in the semiopen model must be less than that of the corresponding open model. This is because the total number of possible local states of a site next to the closest boundary can never exceed 3 ($<\zeta$). However, as far as the thermodynamic behavior of the system is concerned, the difference between the open and semiopen boundary conditions is not that important. In conclusion, all the above discussions on the open boundary models should also work for the semiopen one.

D. Higher-dimensional methods

By direct deduction, we can generalize some of the above results to higher-dimensional models. Without loss of generality, we shall only discuss the open boundary cases with an $n\times n\times\cdots\times n$ grid. For a k-dimensional model, the corresponding toppling matrix will be a kth generation Toeplitz matrix. Hence, the eigenvalue shall be in the form $2k-\sum_{l=1}^{k}2\cos[a_l\pi/(n+1)]$, where $a_l=1,2,\ldots,n$ for each l. Moreover, in the thermodynamic limit, the total number of self-organized critical configurations of the system scales as $(\zeta_k)^{n^k}$, where ζ_k, the average number of distinct possible self-organized critical configurations per site, is given by

$$\ln\frac{\zeta_k}{4}=\left[\frac{2}{\pi}\right]^k\int_0^{\pi/2}\cdots\int_0^{\pi/2}\ln\left[\sum_{i=1}^{k}\sin^2 x_i\right]dx_1\cdots dx_k\ . \tag{12}$$

So for a sufficiently large value of $k,\zeta_k\to 2k$. In other words, the system approaches the trivial case where all the $2k$ particles topple each time they are completely dissipated. In this way, the lower bound of the p_c corresponding to the k-dimensional site percolation model shall approach $1/2k$ for sufficiently large k. This observation is consistent with results from other arguments [16]. Besides, following the same iterative procedure as in Theorem 2 we can find the unitary matrix that diagonalizes Δ and hence the two-point correlation function. Its explicit form is

$$G_{i_1,\ldots,i_k;j_1,\ldots,j_k}=\left[\frac{2}{n+1}\right]^k\sum_l\frac{\prod_l\left[\sin\frac{i_la_l\pi}{n+1}\sin\frac{j_la_l\pi}{n+1}\right]}{2k-2\sum_l\cos\frac{a_l\pi}{n+1}}\ , \tag{13}$$

where the sum is over all a_l from 1 to n with $l=1,2,\ldots,k$. In fact, a sum rule similar to Eq. (10) can also be constructed. Again a natural consequence of the sum rule is that the avalanche size cannot scale like $1/s^{\alpha}$. Let us consider also the sum

$$\sum_{i_l}G_{1,\ldots,1;i_1,\ldots,i_k}\approx\frac{1}{\pi^k}\int_0^{\pi}\cdots\int_0^{\pi}\frac{\prod_l(1+\cos x_l)}{2k-2\sum_l\cos x_l}dx_1\cdots dx_k\ , \tag{14}$$

which is finite and nonzero whenever $k\geq 3$, even if the integrand has a pole at $x_1=\cdots=x_k=0$. Thus, the correlation between two distant grid points in the system may either be algebraic or exponentiallike. Using arguments similar to those in the previous subsection, we conclude that the power spectrum of the avalanche size of the system is in the form $1/(f+f_0)^2$ for some $f_0 \geq 0$ (in general, f_0 is also a function of dimension k). In order to calculate the value of f_0, let us consider the following sum (here the role of s is to estimate the value of f_0; in fact, the sum below is finite whenever $f_0 > s$ and is infinite whenever $f_0 < s$):

$$\sum_{i_l}\left[G_{1,\ldots,1;i_1,\ldots,i_k}\exp\left[s\sum_l i_l\right]\right]$$

$$> s\sum_{i_l}\left[G_{1,\ldots,1;i_1,\ldots,i_k}\sum_l i_l\right]$$

$$\approx\left[\frac{2}{n+1}\right]^k\sum_{i_l,a_l}\frac{\left[\sum_l i_l\right]\left[\prod_l \sin\frac{a_l\pi}{n+1}\sin\frac{i_l a_l\pi}{n+1}\right]}{2k-2\sum_l\cos\frac{a_l}{n+1}}$$

$$\approx s(n+1)\ , \qquad (15)$$

which diverges for any $s>0$. As a result the two-point correlation function cannot decay exponentially fast. So f_0 should be 0. In other words, the power spectrum in avalanche size in the higher-dimensional models also scales as $1/f^2$. Actually it is not totally unexpected for the value of a to be 0. The physical reason is that except for the system size and the unit size of each grid point, there is no natural length scale and hence time scale in the system [1]. The observation of the $1/f^2$ spectrum is consistent with earlier claims [4], too.

III. SUMMARY

So far, we have explicitly calculated the eigenvalues of the Abelian sandpile model with open boundary conditions in any dimension. Thus we know that the total number of distinct self-organized critical configurations (or we may term it as the eventual phase-space volume of the system under its own dynamics) increases like a power law. However, due to the conservative nature of the model, one of the eigenvalues will tend to zero in the thermodynamic limit. This leads to the long-range correlation effects of the system as reflected by the two-point correlation function. Besides, we have discovered a sum rule in the two-point correlation function and hence we have concluded that the power spectra in avalanche size for the Abelian sandpile model shall scale like $1/f^2$ for all finite dimensions. Unlike all the previous arguments [4], which are based on statistical arguments, ours use only straightforward calculations for the two-point correlation functions. In fact, the method we use in calculating both Δ and G is rather general. After suitable modification, it can also be applied to various other ASM models on different lattices.

Finally, we have compared the problem of self-organized criticality with the problem of site percolation. We have argued that the value of $1/\zeta$ sets a lower bound for p_c in the corresponding site percolation model. In this way, we have offered a new way for calculating either analytically or numerically the lower bounds of critical coverage probabilities in various site percolation models. Further discussions on this issue can be found elsewhere [15].

ACKNOWLEDGMENTS

This work is supported partially by the UPGC grant of Hong Kong and NSF Grant No. PHY-9100283. In addition, one of the authors (H.F.C.) would like to thank the Croucher Foundation for financial support during his stay in Hong Kong. Finally, the authors would like to thank Dr. P. M. Hui for his interesting discussions and the valuable comments of Professor D. Pines on percolation.

APPENDIX A

Proof of Theorem 1. Similar to Fact 2, the characteristic polynomial $f_n(\lambda)$ of $\mathbf{S}_1^n$ is given by

$$f_n(\lambda)=\frac{\sin(n+1)x-p\sin nx}{\sin x}\ , \qquad (A1)$$

where $\lambda=a-2\cos x$ and $p=a-b>0$. So the eigenvalues are just zeros of Eq. (A1).

Case (i). If $p\leq 1$, then consider $x\in[k\pi/n,(k+1)\pi/n]$ for some $k=0,1,\ldots,n-1$. We may further assume that $g(x)=p\sin nx$ is non-negative in this interval; otherwise, the argument is just similar. By direct checking, it is clear that $h(x)=\sin(n+1)x$ attains a maximum value of 1 ($\geq p$) in this interval. Also, we can find x in this interval such that $h(x)$ becomes negative. By continuity of both $g(x)$ and $h(x)$, they must meet at least once in this interval. Moreover, it is not too difficult to see that they meet twice in the intervals where $k=0$ and $n-1$, respectively. So from Eq. (A1) and the fact that there are totally n eigenvalues, each interval, except for $k=0$ and $n-1$, contains exactly one zero for the equation $g(x)=h(x)$ and hence our assertion is obviously satisfied.

Case (ii). If $p>1$, then $\sin(n+1)x=p\sin nx$ can be rewritten as $\sin x=(p-\cos x)\tan nx$, with $p-\cos x$ being always positive. Thus the roots of this equation in the interval $(0,\pi)$ must lie in the region where $\tan nx\geq 0$. That is, $x\in[k\pi/n,(k+0.5)\pi/n]$ for some $k=0,1,\ldots,n-1$. Moreover, due to the continuity of $\tan nx$ and the fact that it runs from 0 to ∞ in this region, there exists at least one root in each of the close intervals. From Eq. (A1) and the fact that f_n has exactly n zeros again, our assertion is also true.

APPENDIX B

Calculation of G. We consider the case where $m=n$. Following the idea of Theorem 2, together with Fact 2, we know that, if $D_{i,j;k,l}$ denotes the matrix

$$\delta_{i,k}\delta_{j,l}(4-2\cos[i\pi/(n+1)]-2\cos[j\pi/(n+1)])\ ,$$

we can also construct the unitary matrix $\mathbf{V}$ such that $\mathbf{VDV}^{-1}$ is block tridiagonalized and hence

$$(\mathbf{V}^{-1}\mathbf{D}^{-1}\mathbf{V})_{i,j;k,l}=\frac{2\delta_{i,k}}{n+1}\sum_{a=1}^{n}\frac{\sin\frac{ja\pi}{n+1}\sin\frac{la\pi}{n+1}}{4-2\cos\frac{i\pi}{n+1}-2\cos\frac{a\pi}{n+1}} . \tag{B1}$$

If we interchange i with j and k with l, the matrix $\mathbf{VDV}^{-1}$ will be block-diagonalized, with each block being tridiagonal. So by applying the same trick again, it is not difficult to see that the same $\mathbf{V}$ as before can diagonalize this block-diagonal matrix. So we have

$$G_{i,j;k,l}=\sum_{a,b,c,d}\mathbf{V}^{-1}_{i,j;a,b}(\mathbf{V}^{-1}\mathbf{D}^{-1}\mathbf{V})_{b,a;d,c}\mathbf{V}_{c,d;k,l} . \tag{B2}$$

By direct calculation, Eq. (9) is proved. Following the same argument, Eq. (9) can also be generalized to higher dimensions.

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